Superficial Tableaux for Contextual Reasoning*

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Abstract

This paper presents a tableaux calculus for the Propositional Logic of Contexts with the $\mathtt{ist}(c, \phi)$ modality. This approach has a twofold advantage: from the user viewpoint it presents rules which intuitively reflect epistemic properties (lifting, use of assumptions etc.); from a computational perspective it allows local and incremental computation, satisfies strong confluence and can therefore be adapted efficiently to different search heuristics. The modelling of contexts as partial objects is obtained by using superficial assignments. We can define meaningful and meaningless sentences and reason about formulae containing both kind of sentences. Superficial valuations provide us with a sound and incremental approximation of classical logic and make it possible to present a simplified semantics based on layered models.

Introduction

The relevance of contextual reasoning for Artificial Intelligence and Computer Science has been firstly proposed by McCarthy in his Turing Award Lecture (1971) and recentely discussed in (McCarthy 1993). The need to put theories in their contexts could also be traced back to meta-programming and meta-reasoning (Aiello & Levi 1984), while in (Fagin & Halpern 1988) the modeling of human reasoning as non interacting clusters (contexts) has been advocated for modeling limited omniscience.

Since these seminal papers, there have been many proposals to formalize contextual reasoning, some motivated by the need to provide a theoretical ground to implemented systems. Just to mention a few examples, we may start with the proposal of (Shoham 1991), to introduce within the language an exponent ϕ^c when ϕ holds in context c, or the formalization of (Guha 1991) where the expression $\mathtt{ist}(c, \phi)$ is used, an approach followed to develop the Propositional Logic of Contexts in (Buvač & Mason 1993; Buvač, Buvač, & Mason 1995); one may also think of Multi-Contextual and Multi-Languages systems developed in (Giunchiglia *et al.* 1993; Giunchiglia & Serafini 1994) for limited omniscience and meta-reasoning, where $\langle \phi, i \rangle$ means that ϕ holds in the i-th meta-theory. A further step towards meta-reasoning is developed by the theory of viewpoints in (Attardi & Simi 1995), where $\mathtt{in}('\phi', vp)$ is used. A similar approach has been recently pursued in the field of modal logics if one substitutes contexts for possible worlds. For instance, we may consider the Labelled Deductive Systems (Gabbay 1994) for modal and sub-structural logics or the prefixed tableaux for modal logics (Fitting 1983).

Although arising from different perspectives, all proposals share a common intuition: *contextualize formulae*, i.e. label formulae with additional information to "locate" them. These locations can be sequences of contexts, possible worlds, stages of the derivation process etc. according to one's logic.

Beyond the definition and formalization of a logical system (by Hilbert or natural deduction calculi) one wants to perform some automated reasoning. This requires to develop calculi like resolution or tableaux.

In parallel with the idea of contextualizing formulae, a new approach has been developed for theorem proving: *labelled deduction*, i.e. deduction using rules which take into account formulae *and* labels. Intuitively we may see operations on labels as a way to "move" formulae from one context, possible world etc. to another. For instance consider the context lifting rules in (McCarthy 1993), the bridge rules for meta-reasoning and communication in (Giunchiglia & Serafini 1994), the reflection rule for meta-reasoning in (Attardi & Simi 1995), the single step modal rules in (Massacci 1994), or the visa rule of (Gabbay 1994).

However, a classical system is not enough: it has been often argued that contexts should be *partial objects*, either because the description of the world may be only partial, or because something meaningful in context may well be meaningless in another. Different approaches have been proposed: meaningless is interpreted as falsity in (Guha 1991), whereas Bochvar three valued logic is used in (Buvač, Buvač, & Mason 1995). In (Giunchiglia & Serafini 1994) different forms

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of incompleteness (in the language, in the deduction or formation rules) are discussed.

In this paper we merge these three approaches (contextualized formulae, labelled deduction, partial information) to develop a Superficial Tableaux Calculus for the Propositional Logic of Context with the $\mathtt{ist}(c, \phi)$ modality introduced by (Guha 1991).

As a tool for automated reasoning it has many advantages: from the point of view of a "lay" user and from a computational perspective. Indeed a user is provided with rules intuitively representing the communication properties (lifting, database assumptions etc.) s/he may be interested to model. The use of contextualized formulae makes it simpler to construct a counter model for non valid formulae. From a computational viewpoint it allows us to process formulae one by one and thus an incremental deduction which is essential for large databases. It also satisfies the strong confluence property and therefore can be adapted to many search heuristic. Finally, a completeness theorem is fairly simple.

To represent partial information, we propose an approach based on the superficial and *shallow valuation*[1] proposed by Levesque in (Levesque 1984) and further extended and studied in (Schaerf & Cadoli 1995). The basic intuition is that superficial valuations assign opposite truth value to a meaningful sentence and to its negation, whereas meaningless sentences are always interpreted as satisfiable (i.e. contradiction are not detected). This approach has many advantages (Schaerf & Cadoli 1995): it is a sound but incomplete approximation of classical logic; yet we can tune the degree of approximation by increasing the vocabulary of meaningful sentences.

In the next section we present the intuition underpinning superficial tableaux and give a flavor of the features of the systems. Afterwards we introduce syntax and semantics and successively present the tableaux calculus with an example of a tableau proof. Its computational properties and the completeness of the calculus are discussed in the subsequent section. Finally we discuss related work and conclude.

Intuitions

We represents properties of contexts with the language of the propositional logic of contexts introduced by (Guha 1991). Thus we build formulae from the set of propositional letters with propositional connectives $\neg$, $\wedge$, $\rightarrow$ etc. and the contextual modality $\mathtt{ist}(c, \phi)$, where c is a context and ϕ is a formula. Intuitively $\mathtt{ist}(c, \phi)$ means that ϕ holds in context c. Here, we follow (Buvač, Buvač, & Mason 1995) and use the interpretation of $\mathtt{ist}()$ as "is valid" in a context. Thus a context c can be seen as a set of interpretations and $\mathtt{ist}(c, \phi)$ means that ϕ must be true in all interpretations of c. However, it is worth noting that $\mathtt{ist}()$ has also been interpreted as **is true** in a context (that is a context as one interpretation).

Such a powerful language makes it possible to express that contexts may be themselves context dependent. For instance, we can represent that in the Hawaii Islands, sharks are dangerous for fisherman with:

$$\mathtt{ist}(hawaii, \mathtt{ist}(fisherman, \text{sharks} \rightarrow \text{danger})) \quad (1)$$

However, since our objective is automated reasoning, we still have to contextualize formulae to make them suitable for labelled deduction (with tableaux). For instance, suppose that we are willing to prove that (1) is satisfiable in *this* context. To construct the proof, we may start assuming that there must be one interpretation, say *this* [1], of *this* context where this formula holds. Hence we can contextualize the formula (1) as follows:

$$this\,[1] : \mathtt{ist}(hawaii, \mathtt{ist}(fisherman, \text{sharks} \rightarrow \text{danger})) \quad (2)$$

We may also express that in *this* [1] interpretation it is not always the case that fishermen from Hawaii swim:

$$this\,[1] : \neg\mathtt{ist}(hawaii, \mathtt{ist}(fisherman, \text{swim})) \quad (3)$$

Now we can further expand (3) in a tableaux like fashion (breaking formulae top-down). Intuitively, for (3) to hold, there must a interpretation of the context *hawaii* where our fisherman does not swim. Therefore we can bring this negative information from *this* context to the the context of *hawaii* (within the perspective of *this* context):

$$\begin{array}{c} this\,[1] : \neg\mathtt{ist}(hawaii, \mathtt{ist}(fisherman, \text{swim})) \\ \Downarrow \\ this \cdot hawaii\,[2] : \neg\mathtt{ist}(fisherman, \text{swim}) \end{array}$$

Following (McCarthy 1993), we call this process *lifting*. In this case we lift a negative information (negative lifting) from the *this* context to $this \cdot hawaii$ context. We can iterate this process to get:

$$this \cdot hawaii \cdot fisherman\,[3] : \neg\,\text{swim}$$

At this stage we may reconsider (2): since $\mathtt{ist}()$ represent validity then it must be the case that, also in $this \cdot hawaii\,[1]$ sequence of contexts, fishermen fear sharks. Here we lift the positive information represented by (2) from *this* [1] interpretation to $this \cdot hawaii\,[2]$ interpretation to obtain:

$$this \cdot hawaii\,[2] : \mathtt{ist}(fisherman, \text{sharks} \rightarrow \text{danger})$$

Finally we repeat again positive lifting up to $this \cdot hawaii \cdot fisherman\,[3]$ to obtain :

$$this \cdot hawaii \cdot fisherman\,[3] : \text{sharks} \rightarrow \text{danger}$$

It is worth noting how the deduction process itself explicitly represents the fact that contexts are themselves context dependent.

[1]The name superficial tableaux is due to this characteristic.

What is still to be done is to treat contexts as partial objects. The basic intuition is to introduce a vocabulary $\texttt{vocab}()$ which identifies the atomic sentences (propositional letters) meaningful in a context or better in a sequence of contexts. Indeed, in real situations, it is clear that $\texttt{vocab}(\mathit{this} \cdot \mathit{hawaii} \cdot \mathit{fisherman})$ will be different from $\texttt{vocab}(\mathit{this} \cdot \mathit{italy} \cdot \mathit{fisherman})$ although in both cases the last context is always about *fisherman*. However, to keep things simple, we impose that the vocabulary depends only on the sequences of contexts and not on the particular interpretation. Thus $\texttt{vocab}(\mathit{this}[1]) = \texttt{vocab}(\mathit{this}[2]) = \texttt{vocab}(\mathit{this})$.

Suppose that our fisherman heard two yuppy tourists talking about *insider* trading (which is not in his vocabulary). The key point is how do we evaluate the following contextualized formulae:

$$\mathit{this} \cdot \mathit{hawaii} \cdot \mathit{fisherman}[1] : \text{insider} \vee \neg\text{insider}$$
$$\mathit{this} \cdot \mathit{hawaii} \cdot \mathit{fisherman}[1] : \text{insider} \wedge \neg\text{insider}$$

From the viewpoint of our fisherman these are *both completely meaningless sentences*. Thus we want them to be *both satisfiable* (we assume that speakers are making sense) and *both non valid* (after all we do not understand them).

Although a classical logician may shudder, this is exactly what often happens with common-sense reasoning. For instance consider a person browsing a dictionary (Schaerf & Cadoli 1995): if we do not understand a sentence in an entry, we surely do not start checking whether it is a propositional tautology or a logical consequence of some previous definitions. On the contrary we assume that this sentence should hold in some context but, since we do not understand it, we just ignore it and move on, concentrating on meaningful sentences.

To capture this behavior with a firmer theoretical framework we make use of superficial evaluation w.r.t. the vocabulary, extending the work of (Levesque 1984; Schaerf & Cadoli 1995). Intuitively, if a propositional letter p is not in the vocabulary of a context c i.e. $p \notin \texttt{vocab}(c)$ then every superficial valuation σ is such that $\sigma(p) = \sigma(\neg p) = \mathit{true}$. Only if $p \in \texttt{vocab}(c)$ we impose that $\sigma(p) \neq \sigma(\neg p)$ i.e. only meaningful sentences "deserve" a two valued interpretation.

Therefore the classical tableau rule for discarding inconsistency is simply updated: discard a branch only if one finds contradictory literals w.r.t. the vocabulary of the context in which they are supposed to hold. For instance, in the case of the fisherman

$$\mathit{this} \cdot \mathit{hawaii} \cdot \mathit{fisherman}[1] : \text{sharks}$$
$$\mathit{this} \cdot \mathit{hawaii} \cdot \mathit{fisherman}[1] : \neg\,\text{sharks}$$

could lead to a inconsistent branch since we assume that *sharks* is part of the vocabulary. However the contradiction would not have been detected if we replaced *sharks* with *insider*.

It is also worth noting the differences with multi valued logics since they, as classical logic, "give the same chances" to all propositional variables, i.e. the set of truth values that a particular propositional letter can assume is not restricted a priori and does not vary from letter to letter: all propositional letter can assume all n (or infinite) truth values. Therefore they also are not sensible to the context. For instance Kripke/Kleene 3-valued logic does not have propositional tautologies. In contrast, superficial valuations still has tautologies, although restricted to meaningful sentences. Thus the set of propositional tautologies varies according the context (or better according the vocabulary of the context).

Superficial valuations make it also possible to reason about formulae mixing meaningful and meaningless sentences. Indeed meaningless propositional letters do not affect the closure of the tableau and thus we can interpret *meaningless sentences as noise in the communication*. If it is not too much (i.e. it doesn't create branches only with completely meaningless sentences) we can still make sensible deductions.

By switching context we change the vocabulary of meaningful sentences (assuming $\texttt{vocab}()$ is not a constant function) and thus the deductions we can make. In some cases we may use this property to simulate approximate classical reasoning: increase iteratively the vocabulary till it contains all propositional letters.

Syntax and Semantics

We assume some basic knowledge of the formalism developed for the *Propositional Logic of Context* of (Guha 1991; Buvač & Mason 1993; McCarthy 1993; Buvač, Buvač, & Mason 1995). Thus if $\mathcal{P}$ is a set of propositional letters and $\mathcal{C}$ is a set of contexts the *language* $\mathcal{L}_{\text{CXT}}$ is the least set such that:

1. $\mathcal{P} \subset \mathcal{L}_{\text{CXT}}$
2. if $\psi, \phi \in \mathcal{L}_{\text{CXT}}$ then $\neg\phi, \phi \wedge \psi \in \mathcal{L}_{\text{CXT}}$
3. if $c \in \mathcal{C}$ and $\phi \in \mathcal{L}_{\text{CXT}}$ then $\texttt{ist}(c, \phi) \in \mathcal{L}_{\text{CXT}}$

Other connectives can be seen as abbreviations. In the sequel ϕ, ψ are formulae whereas c (eventually with indices) is a context. The set of sequences over $\mathcal{C}$ is denoted by $\mathcal{C}^*$ whereas a sequences of contexts $\langle c_1, c_2 \ldots c_n \rangle$ by $\boldsymbol{c}$. In the sequel $\boldsymbol{c} \cdot c_i$ is the usual concatenation of $\boldsymbol{c}$ and $\langle c_i \rangle$. We also say that a sequence $\boldsymbol{c}^*$ *extends* a sequence $\boldsymbol{c}$ if there is another sequence $\boldsymbol{c}^\bullet$ such that $\boldsymbol{c}^* = \boldsymbol{c} \cdot \boldsymbol{c}^\bullet$.

Each application (and the corresponding tableau) has its *vocabulary*, i.e. a function $\texttt{vocab}() : \mathcal{C}^* \to 2^{\mathcal{P}}$ from sequences of contexts to sets of propositional letters. As already noted, only the propositional letters $p \in \texttt{vocab}(\boldsymbol{c})$ will receive a careful evaluation (i.e. a 2-valuation) whereas if p is not in $\texttt{vocab}(\boldsymbol{c})$ then it is meaningless and a superficial evaluation will be fair enough. This is very close to the notion of $3 - S$-valuation of (Schaerf & Cadoli 1995) where S is $\texttt{vocab}(\boldsymbol{c})$.

Definition 1 *Let $V \subseteq \mathcal{P}$ be a set of propositional letters, then a* superficial valuation *w.r.t. V is a function $\sigma_V : \mathcal{P} \cup \neg\mathcal{P} \rightarrow \{true, false\}$ such that*

- *$\sigma_V(p) \neq \sigma_V(\neg p)$ for any $p \in V$,*
- *$\sigma_V(p) = \sigma_V(\neg p) = true$ for any $p \in \mathcal{P}/V$.*

In the sequel we indicate with $\mathcal{S}_V$ the set of all superficial valuations σ_V w.r.t. V.

Definition 2 *A* layered model *w.r.t. a given vocabulary* $\texttt{vocab}()$ *is a pair $\langle\Sigma, \texttt{vocab}()\rangle$ where Σ is a function which maps a context sequence $\boldsymbol{c}$ into a set of superficial valuations $\sigma_{\texttt{vocab}(\boldsymbol{c})}$ w.r.t. $\texttt{vocab}(\boldsymbol{c})$.*

Intuitively $\texttt{vocab}(\boldsymbol{c})$ gives us the vocabulary (the set of meaningful sentences) of $\boldsymbol{c}$ whereas $\Sigma(\boldsymbol{c}) \subseteq \mathcal{S}_{\texttt{vocab}(\boldsymbol{c})}$ reflects the (superficial) truth valuations which describe that particular $\boldsymbol{c}$. The term layered has been used since we can see a sequence of contexts as a sequence of layers, each layer build up from the superficial valuations proper of that sequence. In the sequel, for sake of readability, we drop the subscript $\texttt{vocab}(\boldsymbol{c})$ from σ, with the understanding that whenever $\sigma \in \Sigma(\boldsymbol{c})$ then this σ is indeed a $\sigma_{\texttt{vocab}(\boldsymbol{c})}$.

Definition 3 *Let $\langle\Sigma, \texttt{vocab}()\rangle$ be a layered model, $\boldsymbol{c}$ a sequence of contexts and $\sigma \in \Sigma(\boldsymbol{c})$ a superficial evaluation, then the entailment relation $\sigma \Vdash_{\boldsymbol{c}} \phi$ between formulae and superficial valuations is defined as follow:*

$\sigma \Vdash_{\boldsymbol{c}} l$	*iff*	$\sigma(l) = true$ *where* $l \in \mathcal{P} \cup \neg\mathcal{P}$
$\sigma \Vdash_{\boldsymbol{c}} \phi \wedge \psi$	*iff*	$\sigma \Vdash_{\boldsymbol{c}} \phi$ *and* $\sigma \Vdash_{\boldsymbol{c}} \psi$
$\sigma \Vdash_{\boldsymbol{c}} \neg(\phi \wedge \psi)$	*iff*	$\sigma \Vdash_{\boldsymbol{c}} \neg\phi$ *or* $\sigma \Vdash_{\boldsymbol{c}} \neg\psi$
$\sigma \Vdash_{\boldsymbol{c}} \neg\neg\phi$	*iff*	$\sigma \Vdash_{\boldsymbol{c}} \phi$
$\sigma \Vdash_{\boldsymbol{c}} \texttt{ist}(c, \phi)$	*iff*	*for every* $\sigma^* \in \Sigma(\boldsymbol{c} \cdot c)$ *it is* $\sigma^* \Vdash_{\boldsymbol{c} \cdot c} \phi$
$\sigma \Vdash_{\boldsymbol{c}} \neg\texttt{ist}(c, \phi)$	*iff*	*there is a* $\sigma^* \in \Sigma(\boldsymbol{c} \cdot c)$ *such that* $\sigma^* \Vdash_{\boldsymbol{c} \cdot c} \neg\phi$.

It is worth noting again that the classical property $\sigma \Vdash_{\boldsymbol{c}} \neg\phi$ if and only if $\sigma \not\Vdash_{\boldsymbol{c}} \phi$ *does not hold* since ϕ may contain meaningless sentences.

Superficial valuations make it possible to simplify the original semantics from (Buvač, Buvač, & Mason 1995) and to assign a semantics to formulae containing meaningful and meaningless sentences together, although only a "superficial semantics"...

Two other concepts are necessary for entailment, with a close similarity to local and global assumptions for modal logic from (Fitting 1983; Massacci 1994):

- the *Local Contextual Database* (for short LB) is a set of formulae holding in the initial context $\boldsymbol{c}$
- the *Global Contextual Database* (for short GB) contains the formulae holding in every contexts sequence $\boldsymbol{c}^*$ extending the initial sequence $\boldsymbol{c}$

Definition 4 *A formula ϕ is* satisfiable *for the contexts sequence $\boldsymbol{c}$ w.r.t. the local and global contextual database, resp. LB and GB, and the vocabulary* $\texttt{vocab}()$, *if and only if there is a layered model $\langle\Sigma, \texttt{vocab}()\rangle$ such that*

- *for every $\boldsymbol{c}^* \in \mathcal{C}$ extending $\boldsymbol{c}$ and every $\sigma \in \Sigma(\boldsymbol{c}^*)$ it is $\sigma \Vdash_{\boldsymbol{c}^*} GB$*
- *there is some $\sigma \in \Sigma(\boldsymbol{c})$ such that $\sigma \Vdash_{\boldsymbol{c}} LB$ and $\sigma \Vdash_{\boldsymbol{c}} \phi$.*

Where $\sigma \Vdash_{\boldsymbol{c}^*} S$ is a shortcut for $\forall\psi \in S : \sigma \Vdash_{\boldsymbol{c}^*} \psi$.

Definition 5 *A formula $\phi \in \mathcal{L}_{\mathrm{CXT}}$ is* superficially valid *for the contexts sequence $\boldsymbol{c}$ w.r.t. the local and global contextual database LB and GB and the vocabulary* $\texttt{vocab}()$ *if and only if $\neg\phi$ is not satisfiable.*

Note that superficial validity is explicitly defined in term of lack of counterexamples, in the spirit of superficial entailment: a sentence ϕ should be (superficially) valid if we cannot provide an evidence of the contrary i.e. if we cannot find a model where $\neg\phi$ holds, given our local and global premises. In classical logic one usually defines ϕ as valid iff it holds in every model and then shows that if ϕ holds everywhere then one cannot find a model where $\neg\phi$ holds and viceversa.

However, this is not possible for superficial valuations. Indeed, while meaningful formulae behave in a perfectly classical way (and therefore this property holds for them), meaningless ones do not satisfies it. For instance consider a propositional letter p and a context $\boldsymbol{c}$ such that $p \notin \texttt{vocab}(\boldsymbol{c})$. Then, for every superficial valuation $\sigma_{\texttt{vocab}(\boldsymbol{c})}$ it is *both* $\sigma \Vdash_{\boldsymbol{c}} p$ and $\sigma \Vdash_{\boldsymbol{c}} \neg p$. Still, we would not like to define p as "valid" in this case. Hence, to obtain the desired behavior of meaningless formulae we need to use a definition of superficial validity which explicitly requires the absence of counterexamples. With this modified definition we can be sure that meaningless sentences are always satisfiablebut never superficially valid.

Superficial Tableaux

Superficial tableaux (for a general introduction to tableaux see (Fitting 1990)) are based on the same ideas of (Massacci 1994): labelled formulae and labelled deduction. Here, the labels attached to a formula must capture two semantical information: the sequence of contexts and the superficial interpretation (of that sequence). Therefore a *contextualized formula* is a pair $\langle \boldsymbol{c}[n] : \phi\rangle$ where $\boldsymbol{c}$ is a sequence of contexts, n an integer and ϕ a formula of $\mathcal{L}_{\mathrm{CXT}}$. Intuitively the *contextual prefix* $\boldsymbol{c}[n]$ "names" the n-th superficial valuation of $\Sigma(\boldsymbol{c})$, where ϕ holds.

Then, to prove that ϕ is valid for the context $\boldsymbol{c}$ we try to construct a counter-model i.e. we assume $\langle \boldsymbol{c}[1] : \neg\phi\rangle$ and successively expand the structure according the connectives of ϕ. Therefore a tableaux proof can be seen as a (binary) tree, whose nodes are labelled with contextualized formulae. Tableaux rules transform a tree into another tree, expanding the structure (adding new nodes or branching the tree). At the same time contradictory[2] branches are discarded, since they do not yield a counter model.

[2]Of course contradictory for a superficial valuation.

$$\alpha : \frac{\boldsymbol{c}^*[n] : \phi \wedge \psi}{\begin{array}{c}\boldsymbol{c}^*[n] : \phi \\ \boldsymbol{c}^*[n] : \psi\end{array}} \qquad \beta : \frac{\boldsymbol{c}^*[n] : \neg(\phi \wedge \psi)}{\boldsymbol{c}^*[n] : \neg\phi \quad | \quad \boldsymbol{c}^*[n] : \neg\psi} \qquad \text{dneg} : \frac{\boldsymbol{c}^*[n] : \neg\neg\phi}{\boldsymbol{c}^*[n] : \phi}$$

$$\text{Loc} : \frac{\vdots}{\boldsymbol{c}[1] : \psi} \quad \text{if } \psi \in LB, \text{ where } \boldsymbol{c} \text{ is the initial context}$$

$$\text{Glob} : \frac{\vdots}{\boldsymbol{c}^*[n] : \psi} \quad \text{if } \psi \in GB, \boldsymbol{c}^*[n] \text{ is present and extend the initial contexts sequence } \boldsymbol{c}$$

Figure 1: Propositional and Databases Rules

$$\text{Pos-lifting} : \frac{\boldsymbol{c}^*[n] : \mathtt{ist}(c, \phi)}{\boldsymbol{c}^* \cdot c[m] : \phi} \quad \text{if } \boldsymbol{c}^* \cdot c[m] \text{ is present in the branch}$$

$$\text{Neg-lifting} : \frac{\boldsymbol{c}^*[n] : \neg\mathtt{ist}(c, \phi)}{\boldsymbol{c}^* \cdot c[m] : \neg\phi} \quad \text{if } \boldsymbol{c}^* \cdot c[m] \text{ is new for the branch}$$

Figure 2: Rules for Context Lifting

Hence the definition of tableau and branch is standard (Fitting 1990) but reduction rules and the closure rule. These rules require some more terminology i.e. $\boldsymbol{c}[n]$ is *present in a branch* if, for some ϕ there is a contextualized formula $\langle \boldsymbol{c}[n] : \phi \rangle$ already in the branch, whereas it is *new for a branch* if it is not present.

The rules for propositional connectives are quite standard and are shown in Fig. 1 together with the rules to process formulae from global and local contextual databases.

The "truly" contextual rules are the lifting rules listed in Fig. 2. Their intuitive interpretation is simple, as names themselves suggest: they lift to a context what is said to hold in that context. For instance consider negative lifting: if $\neg\mathtt{ist}(c, \phi)$ holds in a given context $\boldsymbol{c}$ (more precisely in a sequence of contexts) then there must be an interpretation for c where $\neg\phi$ is satisfiable. Hence with negative lifting we give a "name" to this interpretation with $\boldsymbol{c} \cdot c[m]$ and "lift" $\neg\phi$ to this context. Of course this label (name) must be new: we are just assuming it exists without making any other assumption on it.

In this way we have a much simpler proof theory than (Buvač, Buvač, & Mason 1995), where three axioms and one rule are necessary only to cope with $\mathtt{ist}(,)$. Once again, the credit should go to the technique: contextualized formulae and labelled deduction.

The closure rule must be modified w.r.t. standard tableaux: only the contradiction of meaningful sentences for our vocabulary $\mathtt{vocab}()$ does matter. Meaningless sentences may be contradictory but…who cares? After all they are meaningless!

Definition 6 *Let $\mathcal{B}$ be a branch on a tableau then $\mathcal{B}$ is* closed *w.r.t. the vocabulary* $\mathtt{vocab}()$ *if and only if, for some contextual prefix $\boldsymbol{c}^*[n]$ and some $p \in \mathtt{vocab}(\boldsymbol{c}^*)$ there are two contextualized literals $\langle \boldsymbol{c}^*[n] : p \rangle$ and $\langle \boldsymbol{c}^*[n] : \neg p \rangle$ both present in $\mathcal{B}$.*

A tableau is closed w.r.t. a vocabulary $\mathtt{vocab}()$ if every branch is such. *Termination* occurs if all possible rules have been applied. A branch is *open* if it is terminated and not closed and a tableau is open if at least one branch is such. Now we have all the machinery necessary to define a tableau proof.

Definition 7 *Let GB and LB be a global and a local contextual databases, and let* $\mathtt{vocab}()$ *be a vocabulary, then ϕ has a* contextual tableau proof *for the contexts sequences $\boldsymbol{c}$ if and only if the tableau starting with $\langle \boldsymbol{c}[1] : \neg\phi \rangle$ closes w.r.t.* $\mathtt{vocab}()$*.*

The proof of the Δ axiom from (Buvač, Buvač, & Mason 1995) is in Fig. 3, numbers are for references.

The proof starts negating the axiom (0) and breaks down its propositional connectives by repeated applications of the α-rule, thus yielding (1a), (1b) and (1c).

Then we use negative lifting to (1c) to introduce one interpretation for the context $\boldsymbol{c} \cdot c_1$ and lift $\neg\psi$ to it, obtaining (2). Now we positively lift (1a) to get (3).

At this stage me move to this new layer and perform some propositional reasoning i.e. apply the β rule to (3) yielding (4a) and (4b).

Here we move back to the first layer again and lift negatively (1b) to (5) thus introducing a new interpretation $\boldsymbol{c} \cdot c_1[2]$. We continue negative lifting and positive lifting to get (6) from (5) and (7) from (4a).

The first branch closes by the contradiction of (6) and (7), whereas the second for (2) and (4b). Of course

$$
\begin{array}{ll}
(0) & \boldsymbol{c}[1] : \neg(\mathtt{ist}(c_1, \mathtt{ist}(c_2, \phi) \vee \psi) \\
 & \qquad \rightarrow \mathtt{ist}(c_1, \mathtt{ist}(c_2, \phi)) \vee \mathtt{ist}(c_1, \psi)) \\
(1a) & \boldsymbol{c}[1] : \mathtt{ist}(c_1, \mathtt{ist}(c_2, \phi) \vee \psi) \\
(1b) & \boldsymbol{c}[1] : \neg\mathtt{ist}(c_1, \mathtt{ist}(c_2, \phi)) \\
(1c) & \boldsymbol{c}[1] : \neg\mathtt{ist}(c_1, \psi) \\
(2) & \boldsymbol{c} \cdot c_1[1] : \neg\psi \\
(3) & \boldsymbol{c} \cdot c_1[1] : \mathtt{ist}(c_2, \phi) \vee \psi
\end{array}
$$

$$
\begin{array}{ll|ll}
(4a) & \boldsymbol{c} \cdot c_1[1] : \mathtt{ist}(c_2, \phi) & (4b) & \boldsymbol{c} \cdot c_1[1] : \psi \\
(5) & \boldsymbol{c} \cdot c_1[2] : \neg\mathtt{ist}(c_2, \phi) & & \times \\
(6) & \boldsymbol{c} \cdot c_1 \cdot c_2[1] : \neg\phi & & \\
(7) & \boldsymbol{c} \cdot c_1 \cdot c_2[1] : \phi & & \\
 & \times & &
\end{array}
$$

Figure 3: Proof of the Δ axiom

we assume that, for instance, ψ and $\neg\psi$ can be proved contradictory for $\boldsymbol{c} \cdot c_1$ w.r.t. vocab() i.e. that further expansions will yield a closed tableaux. Indeed ϕ and $\neg\phi$ are not necessarily inconsistent, they may just be meaningless sentences for $\mathtt{vocab}(\boldsymbol{c} \cdot c_1 \cdot c_2)$.

Note how the proof itself proceeds also by layers.

Computational Features and Completeness

A first observation is that using databases rules rather than the deduction theorem makes it possible a more compact and intuitive representation of knowledge and also a relevant computational advantage. Indeed, in most applications, many LB formulae are irrelevant for proving ϕ and $|LB| >> |\phi|$. Here we can incrementally call in formulae from LB according heuristics. Stronger considerations apply for GB which could only be replaced by adding to LB an increasing sequence of formulae $\mathtt{ist}(c_1, \psi)$, $\mathtt{ist}(c_2, \mathtt{ist}(c_1, \psi))$ etc.

Another advantage is the *locality principle* i.e. only few data items (closely related) are required for each rule: superficial tableaux reduce formulae one by one, at most requiring access to two adjacent layers (contexts sequences) of formulae. Thus a large amount of data can be left in secondary storage to be retrieved only when necessary.

An important property is *strong confluence*: if at stage of the computation some rules can be applied then one can apply them in any order. This is granted by the following theorem:

Theorem 1 *Superficial tableaux rules are strongly confluent provided* $\mathtt{ist}(c, \phi)$ *formulae can be used more than once.*

Proof. The proof follows a standard argument for term rewriting systems, namely Neuman's theorem. The key point is that one must always reduce the principal connective first and thus critical pairs not involving ist() are avoided. At the same times, since $\mathtt{ist}(c, \phi)$ can be reused, the system is locally confluent. Moreover each metavariable (i.e. $\boldsymbol{c}$, n, ϕ etc) occurs only once on the top side of each rule (left-linearity in term rewriting terminology). □

These two properties make contextual tableaux adaptable to different proof search heuristics (backtracking, best-first…) which can be tuned according the particular application.

Another important property of superficial tableaux is their *monotonicity w.r.t. increases in the vocabulary* i.e. by increasing the number of the meaningful sentences we enhance our deductive capabilities:

Theorem 2 *If the formula* ϕ *has a superficial proof w.r.t. the vocabulary* vocab() *for the databases* LB *and* GB *and the sequence of contexts* $\boldsymbol{c}$*, and for every sequence of contexts* $\boldsymbol{c}^*$ *extending* $\boldsymbol{c}$ *it is* $\mathtt{vocab}(\boldsymbol{c}^*) \subseteq \mathtt{vocab}^\star(\boldsymbol{c}^*)$ *then* ϕ *has a superficial proof w.r.t.* $\mathtt{vocab}^\star()$*.*

Proof. The proof is straightforward by observing that an application of the closure rule w.r.t. vocab() is still valid w.r.t. an enlarged vocabulary $\mathtt{vocab}^\star()$. □

From monotonicity we can prove that superficial tableaux are a *sound approximation of classical logic* for propositional contexts

Corollary 3 *If the formula* ϕ *has a superficial proof for the databases* LB *and* GB *and the sequence of contexts* $\boldsymbol{c}$ *w.r.t. the vocabulary* vocab() *then it has a proof for the total vocabulary* $\lambda\boldsymbol{c}^*.\mathcal{P}$ *(i.e. a classical proof).*

The interplay of meaningful and meaningless sentences expressed by monotonicity can be made more precise. We can thus explain the intuitive affirmation that *meaningless sentences are noise in communication.* For this we need to introduce the concept of completely meaningless formulae for a given context.

Definition 8 *A formula* ψ *is completely meaningless (CML) for a sequence of contexts* $\boldsymbol{c}$ *and the vocabulary* vocab() *iff it has the form*

- p *where* $p \notin \mathtt{vocab}(\boldsymbol{c})$*;*
- $\neg\psi_1$ *or* $\psi_1 \wedge \psi_2$ *where* ψ_1*,* ψ_2 *are CML w.r.t.* $\boldsymbol{c}$ *and* vocab()*;*
- $\mathtt{ist}(c, \psi_1)$ *where* ψ_1 *is CML w.r.t.* $\boldsymbol{c} \cdot c$ *and* vocab()*.*

Then it is easy to prove the following properties:

Proposition 4 *Let* ψ *be a completely meaningless formula w.r.t.* $\boldsymbol{c}$ *and* vocab()*, then it is superficially satisfiable but not valid.*

Proposition 5 *Let* ϕ *be a satisfiable formula for* GB *and* LB *and the sequence of context* $\boldsymbol{c}$ *w.r.t. the vocabulary* vocab() *then, for any formula* ψ *completely meaningless w.r.t.* $\boldsymbol{c}$ *and* vocab()*,* $\phi \wedge \psi$ *and* $\phi \vee \psi$ *are still satisfiable.*

Proposition 6 *Let ϕ be a superficially valid formula for GB and LB and the sequence of context $\boldsymbol{c}$ w.r.t. the vocabulary* $\texttt{vocab}()$ *then, for any formula ψ completely meaningless w.r.t. $\boldsymbol{c}$ and* $\texttt{vocab}()$*, $\phi \vee \psi$ is still superficially valid whereas $\phi \wedge \psi$ is never superficially valid.*

Last but not least superficial assignments can have polynomial entailment in many non-horn cases (see (Schaerf & Cadoli 1995) for a comprehensive analysis). To take advantages of this possibility we need to use a suitable heuristic: the intuition is that we can "forecast" the satisfiability of a formula before completely reducing formulae into literals. For instance we can simply write off a formulae if it does not contain some meaningful literal.

Soundness and completeness are proved by standard arguments for labelled tableaux (one of the reasons for their appeal).

Theorem 7 (Soundness and Completeness) *Let $\boldsymbol{c}$ be a context sequence and GB and LB be a global and local contextual databases, then for any vocabulary* $\texttt{vocab}()$ *a formula ϕ is superficially valid if and only if it has a superficial proof.*

Proof. We sketch the completeness part. The key point is proving that if a tableau does not close we can use an open branch to construct a (counter) model for the initial formula. Indeed an open branch is "saturated" but still (superficially) consistent. This means, for instance, that if $\langle \boldsymbol{c}[n] : \phi \wedge \psi\rangle$ is present in the branch both $\langle \boldsymbol{c}[n] : \phi\rangle$ and $\langle \boldsymbol{c}[n] : \psi\rangle$ are present. Yet for no $p \in \texttt{vocab}(\boldsymbol{c})$ both $\langle \boldsymbol{c}[n] : p\rangle$ and $\langle \boldsymbol{c}[n] : \neg p\rangle$ are present (otherwise the branch will be closed!).

It is then possible to associate a superficial assignment $\sigma(\boldsymbol{c}^\star[n])$ to each contextual prefix $\boldsymbol{c}^\star[n]$ occurring in the branch and superficial assignment. If $p \notin \texttt{vocab}(\boldsymbol{c}^\star)$ then we set $\sigma(\boldsymbol{c}^\star[n])(p) = \sigma(\boldsymbol{c}^\star[n])(\neg p) = \mathit{true}$ whereas if $p \in \texttt{vocab}(\boldsymbol{c}^\star)$ then it is $\sigma(\boldsymbol{c}^\star[n])(l) = \mathit{true}$ iff $\langle \boldsymbol{c}^\star[n] : l\rangle$ is in the branch, where l is either p or $\neg p$. Then Σ maps $\boldsymbol{c}^\star$ into the set of all $\sigma(\boldsymbol{c}^\star[n])$ constructed so far. Thus, by induction on the structure of the formulae, it can be shown that if $\langle \boldsymbol{c}^\star[n] : \phi\rangle$ is present in the branch then $\sigma(\boldsymbol{c}^\star[n]) \Vdash_{\boldsymbol{c}^\star} \phi$.

Another important property is *decidability*:

Theorem 8 *Satisfiability and superficial validity are decidable if the databases GB and LB are finite.*

Proof. If GB is empty it is enough to note that we can introduce a longer sequence of contexts only if the complexity of the labelled formula is strictly decreasing after and therefore we are bound to terminate. If GB is not empty but finite then one can only introduce a subformula of the finite set $GB \cup LB \cup \{\neg\phi\}$ where ϕ is the formula to be proved. Then after some stage we can only repeat the same formuale with a longer prefix. Therefore an algorithm may look for these loops and terminate in any case. $\square$

Related Works and Conclusions

The current research on formalizing contexts may be divided in two main fields (often overlapping): those based on the introduction of a new modality and those based on reification of sentences.

The proposal of (Shoham 1991) to "exponentiate" sentences ϕ^c, can be placed within the first field. In his paper a number of benchmark axioms are presented, each expressing different epistemic properties. Particularly interesting is the introduction of a modal operator $c_1 \bullet\!\!\supset c_2$ to express in the language the fact that c_1 is as general as c_2.

The $\texttt{ist}(c, \phi)$ modality has been proposed by Guha (Guha 1991) to formalize contextual reasoning in the framework of the Cyc project. Context modeled Cyc micro-theories and the problem of meaningless sentences was approached by considering them false by default.

The same modality has been used in (Buvač & Mason 1993; Buvač, Buvač, & Mason 1995) to present an Hilbert system, with various extensions, and a semantics based on partial valuations. One of the feature of this approach was the idea that a context may be itself context dependent: a model should cope with contexts sequences rather than contexts in isolation. The issue of the vocabulary of a context (sequence) is also tackled by using Bochvar three valued logic to cope with meaningless sentences.

Various form of incompleteness (in the signature, in the deduction rules and so on) can be handled by the systems proposed in (Giunchiglia *et al.* 1993). This work is closer to the second field, since it reifies sentences but still uses names for contexts. A key feature of the system is the use of bridge rules for explicit communication between contexts and for meta-reasoning. In (Giunchiglia & Serafini 1994) hierarchies of meta-theories have been further developed for belief and provability.

The meta-reasoning approach has been carried forward also in (Attardi & Simi 1995) by using viewpoints. These can be seen as names (or abbreviations) for theories and make it possible implicit and explicit representation of theories. The system is restricted to 2-valued logic and traditional problems of inconsistency have been avoided by imposing constraints to reflection rules.

In this paper we have proposed a tableaux calculus based on the technique of contextualized formulae and message passing for the Propositional Logic of Contexts with the $\texttt{ist}(,)$ modality, where meaningless is interpreted extending the ideas behind the 3-valued superficial valuation proposed in (Levesque 1984). A simple semantics based on layered models (merging ideas from (Buvač, Buvač, & Mason 1995) and (Schaerf & Cadoli 1995)) has also been given.

The systems we proposed has many advantages: from an epistemic viewpoint and from a computational perspective. Indeed an user is provided with rules intuitively representing the communication properties (lifting, database assumptions etc.) s/he may be interested to model. The use of contextualized formulae make also simpler to build a counter model for non valid formulae. Superficial valuations describe contexts (sequences) with a partial vocabulary and can be used to make sensible, although "superficial", reasoning about formulae containing both meaningful and meaningless sentences. From a computational viewpoint it allows for an incremental deduction and satisfies the strong confluence property and therefore can be adapted to many search heuristic. Moreover the use of superficial tableaux gives us a sound but incomplete approximation of classical logic, such that an user can always tune the degree of approximation by increasing the vocabulary. It can also be extended to take advantages of cases in which entailment is polynomial.

Future research is in the direction of extending this system to cope with the problem of generality and the relation "is more general than" between contexts with rules for the inheritance of knowledge. An extension to first order logic is also planned.

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