A Bridge Between Modal Logics and Contextual Reasoning

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Abstract

The goal of this paper is to present and discuss a simple and rather effective tableau calculus which combines modal logics of knowledge and belief with contextual reasoning.

The system is made by a multiple combination. For modal proofs, it labels formulae as prefixed tableaux but uses message (knowledge) passing rules similar to those of sequent-like tableaux. For contextual deduction, it merges the metalevel information of the labelling system used by Multicontextual Languages with that used by Labelled Deductive Systems. Its semantics is also simple and intuitively based on a property of Kripke models.

The resulting calculus (k-Clusters Tableaux) is effective for automated proofs, applicable to a wide range of modal logics, and adaptable to many search heuristics. It is also easy to use for proof presentation since its rule have intuitive epistemic interpretation (how knowledge and belief can be inherited up and down possible worlds). It is weak enough to satisfy a KB where two consistent modal statements may globally contradict each other (in classical logic), but strong enough to rule out an inconsistent statement.

1 Introduction and Motivations

Modal Logics and Contextual Reasoning are common tools in Artificial Intelligence and Computer Science for knowledge representation and reasoning. Modal Logics are used to model knowledge and belief [9] and recently have been used as a basis for different nonmonotonic formalisms [11]. Contextual reasoning and its relevance for AI has been discussed in [14], and the modelling of human reasoning as non-interacting clusters in [2]. Contextual reasoning has also been advocated to overcome inconsistencies and even as way to get rid of modal logics [6]. The major aim of this paper is to present k-Clusters Tableaux (k-CT for short), a tableau calculus which builds a bridge between these two approaches towards knowledge representation and automated reasoning, and discuss some of its pros and cons.

We claim that effectiveness and rules with a simple epistemic interpretation can be combined in modal logic and that some "real-world" features of contextual reasoning can still be achieved with small changes to the normal framework. Therefore, we combined the features of different approaches toward automated reasoning: k-CT label each formula like prefixed tableaux [3] but use rules similar to those of sequent-like tableaux [4, 8]. Moreover k-CT merge two metalevel information in the labels: those used by Multilanguage systems Languages [6] and those by prefixed tableaux and Labelled Deductive Systems [3, 5]. The result of the "weddings", is (hopefully) a tableau system for modal logics and contextual reasoning which could be computationally effective, epistemologically motivated and human oriented: a system for the "lay reader".

In this paper we sketch the main results whereas details can be found in [13, 12]. In Sect. 2 we give an intuitive presentation of our system. The proof theory and some computational properties are devised in Sect. 3. A well know example from the literature is presented in Sect. 4 to give a flavor of the calculus. Completeness results are in Sect. 5 whereas open questions are in Sect. 6. Related work and conclusion are presented in Sect. 7.

2 From Modal to Contextual Reasoning

The modal part of k-CT is based on a combination of two different tableau systems for modal logics: sequent-like tableaux and prefixed tableaux (for an overview see [3, 4, 8]). More technical details on the modal part can also be found in [13]. The key intuitions behind the modal proofs are quite simple:

- sequences of integers (called prefixes) "name" possible world where formula are supposed to hold;
- the rules of sequent-like and prefixed tableaux are broken down into "smaller steps" (expressed as message (knowledge) passing between possible worlds);
- each logic is characterized by the different "messages" it can pass.

For instance, consider the axiom of positive introspection $\mathbf{K}\phi \rightarrow \mathbf{K}\mathbf{K}\phi$ which is usually referred as (4). In our system it is captured by the following idea: if ϕ is known in a possible world "called" σ, then ϕ must also be known in all possible worlds close to σ (which are named σn, i.e. the string σ plus an n at the end). In symbol this can be expressed by the rule: if $\langle \sigma : \mathbf{K}\phi \rangle$ holds then add $\langle \sigma n : \mathbf{K}\phi \rangle$ for all existing σn. The epistemic interpretation is fairly immediate: (4) allows to inherit knowledge forward. Rules can be collated (in a Hilbert style fashion) to give one's favorite logic. Moreover, being so simple, they satisfy the locality principle (each computation step only requires few data items closely related) and the strong Church-Rosser property. A good basis for a simple, effective, flexible and parallelizable implementation.

The question now is, how to enhance modal logic with contexts?

A proposal to introduce as many accessibility relations as wanted clusters (contexts) has been made in [2]: to conclude that $\mathbf{K}\phi$ is true, one needs to find *some* accessibility relation such that ϕ holds in all possible worlds accessible *through that particular relation*. However, the main objective of [2] was modelling agents with limited capabilities and lack of logical omniscience: sentences like $\mathbf{K}\phi \wedge \mathbf{K}\neg\phi \wedge \neg\mathbf{K}\bot$ become satisfiable in that framework.

If we are interested in local reasoning for its own sake, such system seems too weak. In a context we may believe ϕ (i.e. that $\mathbf{K}\phi$ is true), while we may well affirm that $\mathbf{K}\neg\phi$ is the case in a different context. However, nobody will ever trust someone uttering sentences like $\mathbf{K}\phi \wedge \mathbf{K}\neg\phi \wedge \neg\mathbf{K}\bot$. So, we would like the set $\{\mathbf{K}\phi, \neg\mathbf{K}\phi\}$ to be k-clusters satisfiable for a certain k, but not the sentence $\mathbf{K}\phi \wedge \neg\mathbf{K}\phi$: classically inconsistent sets could be k-clusters satisfiable, whereas inconsistent formulae should be unsatisfiable for any k.

A promising approach towards contextual reasoning and labelled formula has been based on Multilanguage systems (ML) in [7, 6], where contexts are also used to solve the problem of logical omniscience. Contexts are devised by labelling formulae ($\langle\phi, i\rangle$ means that ϕ belongs to the i-th context), and normal rules only affect formulae with the same label $\langle i\rangle$. Special "bridge rules" are used to model a form of meta-reasoning i.e. if $\langle\phi : i\rangle$ holds then $\langle \mathrm{Theor}('\phi') : i+1\rangle$ holds too.

It is worth noting that the structure of prefixes presents a striking difference between ML and LDS. In ML prefixes are essentially flat: they only have the form $\langle n\rangle$. On the contrary, modal prefixes describe a tree starting with the prefix $\langle 1\rangle$ and progressively spanning to $\langle 11\rangle$, $\langle 12\rangle$, $\langle 121\rangle$, $\ldots\langle 1\sigma\rangle$ etc. This is due to the fact that, in modal logic, local assumptions $\psi \in U$ are always added with the starting prefix $\langle 1\rangle$, i.e. $\langle 1 : \psi\rangle$ [3, 13]. New worlds (prefixes) are created from existing ones (named $\langle\sigma\rangle$), by adding a new integer at the end i.e. $\langle\sigma n\rangle$. Thus, it is implicitly assumed that all local assumptions belong to the same context $\langle 1\rangle$. A key observation from [6] is that labels play a different role in the two systems: in ML systems, indices encode the metalevel information about reasoning location (i.e. the context) whereas in prefixed tableaux labels track the metalevel information about the derivation process (i.e. the current possible world).

This is the key for the introduction of contextual reasoning with *k-Clusters Tableaux*: the knowledge location and stage of the derivation process are merged into one. The first integer of a prefix take into account the context and the rest of the prefix describes the current possible world, w.r.t. the initial context. In a nutshell, we pass from a tree (or a flat forest) to a full fledged forest.

The revision of the proof theory is then extremely simple: instead of assigning the local assumptions from U to the same context $\langle 1\rangle$, we split U into k sets, $\Gamma_1, \ldots, \Gamma_k$ and introduce k new rules for dealing with local options: if $\psi \in \Gamma_i$ then add $\langle i : \psi\rangle$. This is enough to provide us with a calculus for k non communicating frames of mind (clusters). All other rules are not changed since they only specify which formulae can be moved between the prefixes $\langle\sigma\rangle$ and $\langle\sigma n\rangle$. Thus, there is no (modal) way to move a formula between two contexts, e.g. from $\langle 1\sigma'\rangle$ to $\langle 2\sigma''\rangle$.

From a semantical viewpoint, even less work is needed: *Kripke frames are just directed graphs*, and graphs have already something corresponding exactly to non interacting frames of mind: just *take a graph with k connected parts (or clusters)*. Then, completeness and correctness are just a matching between a cluster on a graph and a context of a tableau.

3 Proof Theory and Computational Features

Familiarity with basic modal logics is assumed (for an introduction see [10, 9]). We build a formula from propositional letters by the connectives $\wedge, \neg, \mathbf{K}$, where $\mathbf{K}$ stands for knowledge, belief, necessity, etc. and others connective can be seen as abbreviations. In the sequel ϕ and ψ range over modal formulae and, to keep exposition compact, we refer to $\mathcal{L}$-tableaux when referring to tableaux for a logic $\mathcal{L}$, viz K, D, T, KB, K4, K5, K45, KDB, D4, KD5, KD45, B, S4, S5, OM, OB, OK4, OS4, OM$^+$, OB$^+$, OK4$^+$, OS4$^+$.

k-CT use *prefixed formulae* i.e. pairs $\langle\sigma : \phi\rangle$ where σ is a non empty sequence of integers called *prefix* and ϕ is a modal formula. In the sequel, σ, σn and σnm are prefixes. The definitions of branch and tableau are standard and equal (but reduction rules) to those of prefixed tableau in [3]. The definition of closure and termination is simple: a branch is *closed* if there are contradictory prefixed formulae in that branch (i.e. there is a σ such that either $\langle\sigma : \bot\rangle$ or $\langle\sigma : \neg\top\rangle$ are present in the branch, or, for some ϕ, both $\langle\sigma : \phi\rangle$ and $\langle\sigma : \neg\phi\rangle$ are present); a tableau is closed if every branch is closed. *Termination* occurs when no operation is possible. A branch is *open* if it is terminated and not closed. A tableau is open if at least one branch is such.

When carring a modal proof, we have to cope with a set of *global assumptions* G and a set of *local assumptions* U. Intuitively, formulae from G hold on every possible world, whereas formulae from U just in the particular world from which we start to build our model. To devise a k-cluster satisfiability proof, we need to introduce a k*-scattered assignment*. It is a mapping of formulae from U into k sets[1] $\Gamma_1, \ldots \Gamma_k$ such that $U = \bigcup_{i=1}^{k} \Gamma_i$. Intuitively, each Γ_i corresponds to the local formulae which are supposed to hold in the i-th context. In many practical cases, not every assignment will make sense, and the assignments of formulae to contexts will be done by a preprocessor on a semantical basis.

Definition 1. Given global assumptions G, a k*-CT satisfiability proof* in the logic $\mathcal{L}$ for the local assumptions U is the open tableau starting with a k-scattered assignment for U and terminated by using k-CT-rules for $\mathcal{L}$.

k-CT also allow to devise the concept of *local validity*: once an assignment from U to Γ_i has been fixed, it makes sense to inquire if a formula is valid in a particular cluster: to prove that ϕ is valid in the i-th cluster, we just start a tableau with $\langle i : \neg\phi \rangle$ and terminate with a closed tableau.

Propositional connectives are reduced with traditional rules

$$\alpha \;:\; \frac{\sigma : \phi \wedge \psi}{\begin{array}{c}\sigma : \phi \\ \sigma : \psi\end{array}} \qquad \beta \;:\; \frac{\sigma : \neg(\phi \wedge \psi)}{\sigma : \neg\phi \;\mid\; \sigma : \neg\psi} \qquad \text{dneg} \;:\; \frac{\sigma : \neg\neg\phi}{\sigma : \phi}$$

The modal rules which are common to all modal logics. are described below. These add global assumptions from G or introduce a local assumption in the i-th context (once a scattered assignment has been fixed).

$$i - Loc \;:\; \frac{\vdots}{i : \psi} \quad \text{if } \psi \in \Gamma_i, \text{ with } i \in \{1 \ldots k\}$$

$$Glob \;:\; \frac{\vdots}{\sigma : \psi} \quad \text{if } \sigma \text{ is present in the branch and } \psi \in G$$

The negative information represented by $\neg\mathbf{K}\phi$ is reduced with a traditional π-rule, where π stands for possibility. Some more terminology is needed for the π-rule: a prefix is *unrestricted* in a branch ϑ, if it is not the initial part of any other prefix already present in the branch. Intuitively, if ϕ is not believed in the world named by σ (i.e. $\langle \sigma : \neg\mathbf{K}\phi \rangle$ holds), there must be some "close new world" σn where $\neg\phi$ holds. In practice, a π-rule adds a new node (a new possible situation) to the current (tentative) model.

$$\pi \;:\; \frac{\sigma : \neg\mathbf{K}\phi}{\sigma n : \neg\phi} \quad \text{with } \sigma n \text{ unrestricted in the branch}$$

Finally there are the rules which may be used to reduce $\mathbf{K}\phi$-formulae. Those are also called ν-formulae, where ν stands for necessity.

[1] Some may well be empty or not disjoint.

Table 1: Modal Logics and SST-rules

Logic	ν-SST-rules
K	K
KB	K + B
K4	K + 4
K5	K + 4^D + 4^R
K45	K + 4 + 4^R

Logic	ν-SST-rules
T	K + T
B	K + T + B
S4	T + 4
S5	T + 4 + 4^R
KD*	K*-rules + D

Logic	ν-SST-rules
OM	K + T^D
OB	K + T^D + B^D
OK4	K + 4^D
OS4	K + T^D + 4^D
O*+	O*-rules + D

$$K : \frac{\sigma : \mathbf{K}\phi}{\sigma n : \phi} \qquad D : \frac{\sigma : \mathbf{K}\phi}{\sigma : \neg\mathbf{K}\neg\phi} \qquad T : \frac{\sigma : \mathbf{K}\phi}{\sigma : \phi}$$

$$B : \frac{\sigma n : \mathbf{K}\phi}{\sigma : \phi} \qquad 4 : \frac{\sigma : \mathbf{K}\phi}{\sigma n : \mathbf{K}\phi} \qquad 4^R : \frac{\sigma n : \mathbf{K}\phi}{\sigma : \mathbf{K}\phi}$$

$$T^D : \frac{\sigma n : \mathbf{K}\phi}{\sigma n : \phi} \qquad B^D : \frac{\sigma n m : \mathbf{K}\phi}{\sigma n : \phi} \qquad 4^D : \frac{\sigma n : \mathbf{K}\phi}{\sigma n m : \mathbf{K}\phi}$$

Note. σ, σn and $\sigma m n$ must be already present on the branch, i.e. some π-rules must have introduced them before the application of the ν rule

The epistemic interpretation of these rules in term of message passing is quite intuitive and the proof method can also be simply described: move knowledge back and forth (if the logic allows it) on the tree of possible situations (or the forest if one has many contexts). A final user does not have anymore to worry about transitivity, euclidity and so on. The desired logic can be obtained by a suitable collection of rules, with the same flexibility as Hilbert axiomatizations. The correspondence between ν-rules and modal logics is shown in Table 1.

k-CT calculus satifies many properties which make it particularly suited for automation and practical problem solving. For instance, each rule affects "neighbour" prefixes (i.e. only prefixes of the form σ and σn are involved). Hence, we can use an lean data structure based on a tree (or a forest), which could also easily mirrored by a graphical interface. Indeed it is enough to label each node with $\langle\sigma\rangle$ and every direct children with $\langle\sigma n\rangle$. In this way:

- $i - Loc$ rules add formulae at the root of the i-th tree (i.e. the i-th context);
- *Glob*-rule add formulae to any node;
- propositional rules reduce formulae inside a node or branch the tableau;
- π-rules add new leaves;
- ν-rules move formulae up and down the tree.

until an inconsistent node is found in a branch and the branch is closed.

Note that for each reduction we need to use only to a small portion of the graph represented by the prefixed formulae. Thus k-CT obeys the locality principle i.e. *only few data items at a time must be taken into account.* This is an extremely desiderable property, since data can be left in secondary storage to be retrieved only when necessary.

Another advantage is the Church Rosser property: at any point of the computation the order of applicable rules may be commuted in arbitrary fashion. Thus the system is flexible and adaptable to many different search heuristics.

Theorem 1. *The k-CT calculus is strongly confluent up to isomorphic renaming of prefixes.*

An important feature, shown by this proof, is that only ν-formulae must be reusable; the others can be deleted after reduction.

Decidability can also be proved [3, 13].

Theorem 2. *If a tableau branch is infinite, it must contain a "growing chain" of prefixes and, after a finite number of steps, the chain becomes "periodical" i.e. formulae are just repeated with a longer prefix.*

Thus, even if the underlying countermodel is potentially infinite, an algorithm can check for this periodicity and terminate in any case.

4 An Intuitive Example...

To give an intuitive flavour of k-CT we propose an example of local reasoning due to Halpern et al: a politician dealing with the tradeoff between a balanced budget (bb) and an increase in spending (is). S/he can be globally aware that an increase in spending does not bring a balanced budget, i.e. $G = \{\mathbf{K}(is \rightarrow \neg bb)\}$. Yet, s/he may well believe (in different frames of minds!), that it is necessary to have a balanced budget and to greatly increase spending, i.e. $U = \{\mathbf{K}bb, \mathbf{K}is\}$. We also assume that s/he can reject contradictory beliefs (s/he uses at least logic KD).

If we try to solve this problem in normal modal logic (i.e. 1-CT), the tableau starting with $\langle 1 : \mathbf{K}bb\rangle$ and $\langle 1 : \mathbf{K}is\rangle$ closes and thus U is not satisfiable. This is not the case if we pass to 2-clusters tableaux: we can assign $\mathbf{K}bb$ to $\langle 1\rangle$ and $\mathbf{K}is$ to $\langle 2\rangle$. The tableau below,

$$\begin{array}{lll} (1a) & 1 : \mathbf{K}bb & \text{by } 1 - \text{Loc} \\ (1b) & 2 : \mathbf{K}is & \text{by } 2 - \text{Loc} \end{array}$$

is open for KD, and thus U is 2-clusters satisfiable[2].

It's worth noticing the different use of "," w.r.t. classical logic. In classical logic, every set of facts (e.g. a knowledge base) is always considered as a coherent whole and therefore $\{\mathbf{K}\phi, \neg\mathbf{K}\phi\}$ is not distinguishable from $\{\mathbf{K}\phi \wedge \neg\mathbf{K}\phi\}$. k-CT apply this principle to the set of global assumptions but not to the set of local ones. The set of local assumptions is really considered a simple collection of (possibly consistent) facts, and not necessarily a globally coherent whole. Therefore, the previous two sets can be recognized as different ones. In our formalism, passing from "," to "$\wedge$" means that we have definitely committed ourselves to consider two local assumptions in the same context. In this way, k-CT can cope with globally inconsistent sets but are still sensible to the presence of inconsistent statements. Indeed, if

[2] We could also conclude that is unfair to blame politicians for being inconsistent: they are doing perfect mathematical reasoning with k-clusters tableaux...

$\perp \in \Gamma_i$ for some i, the system will always be k-CT unsatistisfiable and thus anything will be locally derivable also in any other context j. Back to our example, k-CT accept that a reasoner may hold beliefs which are globally inconsistent (since different things may apply to different contexts) but require him/her at least to be consistent in each context.

Therefore k-CT can be a good mathematical formalism to describe and reason about the situation described by Fagin & Halpern in [2]:

> People don't focus on all issues simultaneously: thus, when we say "a believes p", we more properly mean that in a certain frame of mind (when a is focussing on the issues that involve p), it is the case that a believes p. Even if a does perfect reasoning with respect to the limited number of issues on which he is focussing in any given frame of mind, he may not put his conclusion together. Indeed, although in each frame of mind agent a may be consistent, the different conclusions a draws in different frames of mind may be inconsistent. [...] Our key observation is that one reason that people hold inconsistent beliefs is that beliefs tend to come in non-interacting clusters. We can almost view an agent as a society of minds, each with its own set (or *cluster*) of beliefs which may contradict each other.

Future work is in the direction of introducing explicit communications between contexts.

5 Completeness

The semantics of modal logic is the usual one, based on *frames* i.e. pairs $\langle W, \mathcal{R}\rangle$ where W is a non empty set (of possible worlds) and $\mathcal{R}$ is a relation over W. Different modal logics are obtained by varying $\mathcal{R}$ [9, 10]. To keep exposition compact, we again refer to $\mathcal{L}$-frames, $\mathcal{L}$-models or $\mathcal{L}$-tableaux etc.

A $\mathcal{L}$-model is thus a triple $\langle W, \mathcal{R}, \Vdash\rangle$ where $\langle W, \mathcal{R}\rangle$ is a $\mathcal{L}$-frame and $\Vdash$ is a relation between possible worlds and modal formulae such that:

$$\begin{array}{lll}
w \Vdash \top & \text{and} & w \nVdash \perp \\
w \Vdash \phi \vee \psi & \text{iff} & w \Vdash \phi \text{ or } w \Vdash \psi \\
w \Vdash \neg\phi & \text{iff} & w \nVdash \phi \\
w \Vdash \mathbf{K}\phi & \text{iff} & \text{for every } w^* \in W, w\mathcal{R}w^* \text{ implies } w^* \Vdash \phi
\end{array}$$

We also need to formalize the intuition of a cluster as set of connected worlds in a Kripke frame.

Definition 2. Let $\mathcal{F} = \langle W, \mathcal{R}\rangle$ be a frame and $\sim$ the reflexive, symmetric and transitive closure of $\mathcal{R}$, a *cluster* is a set of worlds $\mathcal{C} \in \text{Cluster}(\mathcal{F})$ where:

$$\begin{array}{rcl}
\text{Cluster}(\mathcal{F}) & \doteq & \{[w] : w \in W\} \\
{[w]} & \doteq & \{\, v \in W \ : \ w \sim v \,\}
\end{array}$$

Therefore a *k-clusters frame* is just a frame $\mathcal{F}$, where $|\,\text{Cluster}(\mathcal{F})\,| \leq k$. Similarly, a $\mathcal{L}$-model with k clusters is a model based on a $\mathcal{L}$-frame with k clusters. Then definition of k-clusters satisfiability is straightforward:

Definition 3. Given global assumptions G, the local assumptions U are *k-clusters satisfiable* for the logic $\mathcal{L}$ if and only if there is a k-clusters model $\langle W, \mathcal{R}, \Vdash\rangle$ for the logic $\mathcal{L}$ such that:

- for every $w \in W$ it is $w \| \!- G$ and
- there are k cluster representatives $[w_1], [w_2], \ldots [w_k] \in \mathrm{Cluster}(\mathcal{F})$ such that for any $\psi \in U$ there is a w_i, with $i \in \{1 \ldots k\}$ such that $w_i \| \!- \psi$.

The correctness and completeness proof of modal logics, as given in [13], carries over to k-CT with minor changes:

Theorem 3 (Soundness and Completeness). *Given global assumptions G, the local assumptions U are k-clusters satisfiable iff there is a k-CT satisfiability proof for G and U.*

Proof. we just sketch the completeness part. We apply in a systematic way all the rules to G and U (also to devise all relevant scattered assignments). If the algorithm terminates[3] (no rule can be further applied) and the tableau does not close, then any open branch is saturated for the k-CT. We can now define define a suitable relation among prefixes that could be a "carbon copy" of the relation among possible worlds in a frame with k clusters. The final step require to prove that a branch saturated with the k-CT rules is also saturated with the prefixed rules of Fitting, w.r.t. the "carbon copy" relation. Thus it is possible to prove a strong model existence theorem, by building a model with the saturated branch as in [3]. □

The introduction of local validity, once the splitting of U into the various Γ_i is fixed, can be done in the same way:

Theorem 4. *Given global assumptions G and k sets of local option $\Gamma_1 \ldots \Gamma_k$, ϕ is locally valid in the i-th cluster if and only if the k-CT starting with $\langle i : \neg\phi \rangle$ closes.*

One of the major (side)-effects of this approach is that the modal deduction theorem does not hold anymore when one moves from 1-CT to k-CT for $k > 1$. Indeed in normal modal logics we have the following deduction theorem:

Theorem 5. *The set of local assumptions $U \cup \{\neg\phi\}$ is 1-CT unsatisfiable for the logic $\mathcal{L}$ with global assumptions G iff, for some n, the set of local assumptions $U \cup \{\neg\phi\} \cup G \cup \mathbf{K}G \cup \ldots \mathbf{K}^n G$ is 1-CT unsatisfiable for the logic $\mathcal{L}$ with an empty set of global assumption,i.e.*

$$G \vdash_{\mathcal{L}} U \Rightarrow \phi \quad \textit{iff} \quad \vdash_{\mathcal{L}} U \cup G \cup \mathbf{K}G \cup \ldots \cup \mathbf{K}^n G \Rightarrow \phi \ \ \textit{for some } n \ ,$$

where $\mathbf{K}G = \{\mathbf{K}\phi \mid \phi \in G\}$ *and* $\mathbf{K}^{i+1}G = \mathbf{K}(\mathbf{K}^i G)$

Proposition 6. *The deduction theorem does not hold for k-CT with $k > 1$.*

Use a scattered assignment that maps $U \cup \{\neg\phi\}$ into a context and $\mathbf{K}^i G$ into another one.

6 Open Problems

Question 1. Is it really a simpler and more efficient way for contextual reasoning about knowledge and belief?

[3] In case of non termination, by König lemma, there is an infinite branch that never closes. This branch which can be chosen as the saturated one.

The use of a tree as a data structure simplify the time and memory requirement for executing a single step of the computation. It still remains to be proved that this also allows global savings. It is believed that the answer should be yes, since the Church Rosser property allows us to apply sophisticated heuristics, or even apply different techniques (like Davis Putnam) for the propositional part inside single nodes of the tree.

One of the main obstacles is the decidability check, which requires an expensive loop checking[4]. In this case the following conjecture seems to work well in practice: every proof can be restricted to prefixes not longer then $(n + 3)$ where n is the maximum number of nesting of modal operators and b the total number of disjunctive subformulae in $G \cup U$. The $+3$ is due to the fact that some rules (e.g. the 4^R or the "delayed" ones), starts working only for prefixes longer then 2. If this conjecture is true, one could simply use brute force being sure that the whole process could safely stop after $n + 3$ iterations. The gain in term of computation is clear. However, it may be incompatible with K and S4 being in PSPACE.

Question 2. The property that "inconsistency in a context means everything is derivable in another context" is a bug or a feature?

In practice, it depends on the application. Yet, it limits the flexibility of the systems since this property is logically necessary. At the current stage of development for k-CT one cannot have a model with some inconsistent worlds, unless multiple valued semantics is adopted or multiple accessibility relations are devised. One way to overcome it seems devising explicit communication between contexts with bridge rules as in [7, 6]. In this way, by banning communication with inconsistent contexts, one could safely work with the rest. Explicit communications is simply achieved by rules of the form: if $\langle i : \phi\rangle$ is present then add $\langle j : \psi\rangle$ for suitable formulae. The key question is, how to (smoothly) map them them into the semantics?

Question 3. Can k-CT be smoothly lifted to First Order Logic?

A possibility is to use "constant-passing" rule similar to those of [5]. However, the introduction of contexts make things far less clear for variables. For instance, suppose we have a formula $\langle i\sigma : \forall x.\Phi(x)\rangle$: should x be restricted to individuals existing in the i-context or could it roams over constants belonging to different contexts? No clear cut solutions have been devised so far.

7 Related Work and Conclusions

Tableaux and sequents calculi for modal logics can be found in [3, 4, 8]. Their main drawback is the limited efficiency, especially when strong deducibility is at stake for the logics with negative introspection K5, K45, S5. Indeed, cut has been eliminated for K45 by Schwarz, but only for weak deducibility, i.e. without premises whereas the deduction theorem has been used in [8] . A Church-Rosser property, which is useful for flexible or parallel implementations, is also missing. Indirect methods make use of suitable translation into first order logic based on the idea of coding paths of possible worlds (see [15] for a general overview). Yet, this approach requires a rather heavy equational theory or ad hoc unification procedures. Moreover they seem to need a "user-friendly" interface for proof presentation. Prefixed tableaux [3] and LDS [5] stand in between: they add labels to modal formulae and define rules that take into account both formulae and labels. These approaches provide a simple presentation of proofs and proof search. Yet an equational or algebraic theory is needed to deal with labels.

[4]This seems to be a "feature" of every systems for modal logics, excluding maybe the inverse method.

Clusters accessibility relations are proposed in [2] to get rid of the axiom of logical omniscience. The resulting calculus is very weak since sentences like $\mathbf{K}\phi \wedge \mathbf{K}\neg\phi \wedge \neg\mathbf{K}\bot$ become satisfiable. The same problems has been tackled by [7] in the framework of Multilanguage systems (ML). This formalism is more powerful and flexible than traditional ones. It allows various form of meta-reasoning and modal logic can be simulated in it [6].

A different approach is based on paraconsistent logics where inconsistent formulae are admitted [1].

We have proposed a new tableau calculus that allows an analytic, deterministic, reasonably efficient, strongly confluent, uniform and "user friendly" approach to a wide range of normal propositional modal logics: K,T,D, KB, K4, K5, K45, KDB, D4, KD5, KD45, KD45, T, B, S4, S5, OM, OB, OK4, OS4, OM^+, OB^+, OK4^+, OS4^+ and many others. The calculus also provide contextual reasoning to cope with inconsistent belief sets. k-CT are weak enough to describe a knowledge base where two different consistent beliefs may contradict each other (in classical logic), but strong enough to rule out the possibility of an inconsistent statement. k-CT combine the full power of a modal theorem prover with a system of context handling, by merging the metalevel information used in the labelling system from [3, 5, 13] with those from [7, 6]. They provide a mathematical tool to model and reason about human reasoning when the latter is modelled as a set of non interacting frames of mind as in [2].

Therefore, k-Cluster Tableaux build a bridge between two different approaches towards automated deduction.

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