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Contextual Reasoning is NP-complete

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Abstract

The logic of context with the $\mathtt{ist}\,(c,p)$ modality has been proposed by McCarthy as a foundation for contextual reasoning. This paper shows that propositional logic of context is NP-complete and therefore more tractable than multimodal logics or Multi Language hierarchical logics which are PSPACE-complete. This result is given in a proof-theoretical way by providing a tableau calculus, which can be used as a decision procedure for automated reasoning. The computational gap between logic of context and modal logics is analyzed and some indications for the use of either formalisms are drawn on the basis of the tradeoff between compactness of representation and tractability of reasoning.

Introduction

In the last few years there has been a renewed interest in the use of contexts for natural language understanding and knowledge representation form. Indeed, the discussion about contextual reasoning in AI can be traced back to McCarthy's Turing Award Lecture in 1971, and has been recently tackled by Shoham (1991) and McCarthy (1993). The need to put theories in their contexts is also connected to meta-reasoning (Weyhrauch 1980; Aiello & Levi 1984), while (Fagin *et al.* 1995) also discuss the modeling of human reasoning as non interacting contexts.

There has been many proposals to give a logical formalization of contextual reasoning. Just to mention a few, we may start with (Shoham 1991), which uses an exponent ϕ^c to say that ϕ holds in context c; or the logic of context introduced by (Guha 1991a), where the expression $\mathtt{ist}\,(c,\phi)$ is used; this approach is followed also by Buvač et al. (1993; 1995); one may also refer to the Multi Languages systems by Giunchiglia et al. (1993; 1994), where $\langle\phi, i\rangle$ means that ϕ holds in the i-th meta-theory. Although arising from different perspectives, all proposals share a common intuition: *contextualize the formulae of a knowledge base*, i.e. label formulae with additional information to "situate" them in the appropriate context.

The explicit representation of the context, where a property of a common sense knowledge base holds, or an English sentence is uttered, etc. can be a possible solution to the problem of generality in AI (Buvač & Mason 1993; McCarthy 1993) and also make it possible to effectively manage huge knowledge bases by localizing deduction and constraining the search space (Guha 1991a; Giunchiglia *et al.* 1993).

It is important to notice the practical aspect of this research, as noted in (Giunchiglia & Serafini 1994):

> [...] One of our main interest is, in fact, to *provide foundation to the implementation of "intelligent" reasoning systems.*

Thus Multi Language hierarchical logics (ML systems for short) from Giunchiglia et al. (1993; 1994) may be seen as the foundation of the `FOL` and `GETFOL` systems by (Weyhrauch 1980; Giunchiglia 1992), whereas the logic of context of Guha (Guha 1991a) and Buvač et al. (1993; 1995) could play the same role for `Cyc` micro-theories (Guha & Lenat 1990; Guha 1991a).

In this setting, a question is definitely relevant:

Question 1 *Which is the computational complexity of (these formalizations for) contextual reasoning?*

A partial answer has already been given: the equivalence results with modal logics established by (Giunchiglia & Serafini 1994) prove (indirectly) that their *Multi Language hierarchical logics are PSPACE-complete as modal logics* (Ladner 1977; Halpern & Moses 1992). The question was still open for the logic of context since the decidability proof of (Buvač, Buvač, & Mason 1995) gives a NEXPTIME algorithm.

We show that the satisfiability problem for *propositional logic of context is NP-complete* and give a tableau based decision procedure which solve the satisfiability problem in nondeterministic polynomial time $O(|\Phi|^4)$ in the size of the formula Φ, also when multiple contexts are present. Hence *logic of context is more tractable than modal logics* (unless NP=PSPACE), somehow vindicating McCarthy's claim "Modality yes, modal logic no."

Unfortunately, there is trade off between tractability of reasoning and compactness of representation (Cadoli, Donini, & Schaerf 1994; Gogic *et al.* 1995). In rough terms, PSPACE-completeness means that an exponential model can be polynomially represented by a modal formula (Halpern & Moses 1992). Since the logic of context is NP-complete, then it cannot represent such information in compact form.

Back to intelligent systems this means that *a tree of theories can be compactly stored with modal logics (and ML systems) but not with the logic of context.*

This paper analyses further these differences from the viewpoint of the compactness of the representation and shows how a formula can represent a tree in modal logic but not in propositional logic of context. We also show how logic of context can be seen as an approximation of modal logics.

From these results we conclude drawing up some indications to answer the following question:

Question 2 *When should we choose multi modal logics (or ML systems) or the logic of context?*

Propositional Logic of Context

We assume some basic knowledge of the *propositional logic of context* by (Guha 1991a; McCarthy 1993; Buvač, Buvač, & Mason 1995). Thus, if $\mathcal{P}$ is a set of propositional letters p, $\mathcal{C}$ is a set of contexts c (eventually indexed), then our formulae $\varphi, \psi \in \mathcal{L}_{\mathrm{CXT}}$ are

$$\varphi, \psi \quad ::= \quad p \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathtt{ist}\,(c, \varphi)$$

Other connectives can be seen as abbreviations. In the sequel $\vec{c}_n$ is a sequences of contexts $\langle c_1, \ldots, c_n \rangle$. The sequence $\vec{c}_n \circ c_A$ is a shortcut for the concatenation of $\vec{c}_n$ and $\langle c_A \rangle$. A sequence $\vec{c}_n$ *extends* a sequence $\vec{c}_i$ if there is a sequence $\vec{c}_j$ such that $\vec{c}_n = \vec{c}_i \circ \vec{c}_j$.

Intuitively, $\mathtt{ist}\,(c, \varphi)$ means that φ "is true" in the context c. We follow (Buvač, Buvač, & Mason 1995) and interpret $\mathtt{ist}\,()$ as "is valid" in a context since we are interested in deduction in knowledge bases.

The semantics is a slight modification of Kripke model and we present here a simplified version of the original one from (Buvač, Buvač, & Mason 1995):

Definition 1 *A* layered model *is a pair $\langle W, \Sigma \rangle$ where W is a set of propositional valuations (possible worlds) and Σ a function which maps sequences of contexts $\vec{c}_n$ into subsets of W.*

The term "layered" has been used since a sequence of contexts can be seen as a sequence of layers, each build up from the valuations proper of that sequence. For instance Fig. 1 shows the case of Gen. Powell from US politics, where he is considered a conservative although, among republicans, he is regarded as a leftist.

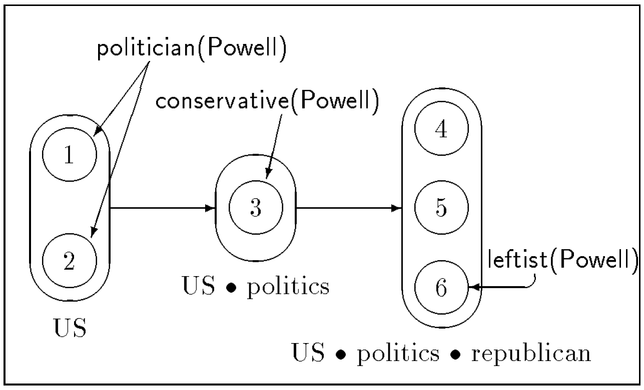

Figure 1: A layered model

The original proposals of (Guha 1991a; Buvač, Buvač, & Mason 1995) also included a vocabulary to cope with partial world descriptions i.e. meaningless sentences (neither true nor false in a given context). In this paper we only deal with complete descriptions. In (Buvač, Buvač, & Mason 1995) this is called *definedness convention.* A tableau calculus which takes into account also meaningless sentences has been developed in (Massacci 1995).

Definition 2 *Let $\langle W, \Sigma \rangle$ be a layered model, $\vec{c}_n$ a sequence of contexts and $w \in \Sigma(\vec{c}_n)$ a valuation, then the entailment relation $\vec{c}_n, w \models \varphi$ is such that:*

- $\vec{c}_n, w \models p$ *iff* $w(p) = \mathtt{true}$ *where* $p \in \mathcal{P}$
- $\vec{c}_n, w \models \varphi \wedge \psi$ *iff* $\vec{c}_n, w \models \varphi$ *and* $\vec{c}_n, w \models \psi$
- $\vec{c}_n, w \models \neg\varphi$ *iff* $\vec{c}_n, w \not\models \varphi$
- $\vec{c}_n, w \models \mathtt{ist}\,(c, \varphi)$ *iff for every* $w^* \in \Sigma(\vec{c}_n \circ c)$ *it is* $\vec{c}_n \circ c, w^* \models \varphi$

Definition 3 *A formula Φ is* satisfiable *for the context sequence $\vec{c}_i$ if and only if there is a layered model $\langle W, \Sigma \rangle$ and some $w \in \Sigma(\vec{c}_i)$ such that $\vec{c}_i, w \models \Phi$.*

An Hilbert axiomatization, under the definedness convention, has been given by (Buvač & Mason 1993; Buvač, Buvač, & Mason 1995) and knowledge bases can be integrated into this framework either with the deduction theorem as in (Buvač, Buvač, & Mason 1995) or directly incorporating them into the tableau calculus as in (Massacci 1995).

Tableau-based Decision Procedure

The tableau calculus – see (Fitting 1990) for an introduction – is based on prefixed tableaux as in (Fitting 1983; Massacci 1994): $\mathcal{L}_{\mathrm{CXT}}$-formulae are labelled with some model theoretic information.

$$\alpha : \frac{\vec{c}_n[m] : \varphi \wedge \psi}{\begin{array}{c}\vec{c}_n[m] : \varphi \\ \vec{c}_n[m] : \psi\end{array}} \qquad \beta : \frac{\vec{c}_n[m] : \neg(\varphi \wedge \psi)}{\vec{c}_n\,[m] : \neg\varphi \quad | \quad \vec{c}_n[m] : \neg\psi} \qquad \text{dneg} : \frac{\vec{c}_n[m] : \neg\neg\varphi}{\vec{c}_n[m] : \varphi}$$

$$\text{Pos-entering} : \frac{\vec{c}_n[m] : \texttt{ist}\,(c, \varphi)}{\vec{c}_n \circ c[k] : \varphi} \quad \text{where } \vec{c}_n \circ c[k] \text{ is present in the branch}$$

$$\text{Neg-entering} : \frac{\vec{c}_n[m] : \neg\texttt{ist}\,(c, \varphi)}{\vec{c}_n \circ c[k] : \neg\varphi} \quad \text{where } \vec{c}_n \circ c[k] \text{ is new and some } \vec{c}_n \circ c[j] : \neg\varphi \text{ does not occur already}$$

Figure 2: Tableau Rules for Contextual Reasoning

For the logic of context, the labels must capture two semantical information: the sequence of contexts and the possible world. So, a *contextualized formula* is a pair $\langle \vec{c}_n[m] : \varphi \rangle$ where $\vec{c}_n$ is a sequence of contexts, m an integer and φ a formula of $\mathcal{L}_{\rm CXT}$. Intuitively, a *contextual prefix* $\vec{c}_n[m]$ "names" the m-th superficial valuation of $\Sigma(\vec{c}_n)$, where φ holds. In Fig. 1 we may represent our knowledge as

$$\langle US, politics\rangle[3] : conservative(Powell)$$

The definition of tableau is standard: a *tableau* is a binary tree, whose nodes are labelled with contextualized formulae, and a *branch* is a path from the root to a leaf. Tableau rules (Fig. 2) transform a tree into another tree by adding nodes or branching the tree. The basic intuition behind a tableau calculus is refutational theorem proving: break down connectives and search for contradictions. If all possible choices lead to a contradiction, then the initial formula is unsatisfiable.

The rules for propositional connectives are standard, whereas the entering rules are the "truly" contextual ones. They require some terminology i.e. $\vec{c}_n[m]$ is *present in a branch* if there is a contextualized formula $\langle \vec{c}_n[m] : \varphi \rangle$ already in the branch, whereas it is *new for a branch* if it is not present. Intuitively, we enter into a context to perform some further deduction there.

For example, we may state that, in the context of US politics, conservative are not leftist and that Powell is a conservative. Still, it should be satisfiable to say that among republicans Powell is not regarded as a leftist:

$$\begin{array}{l} \langle US\rangle[1] : \texttt{ist}(politics, \\ \quad conservative(Powell) \rightarrow \neg leftist(Powell)) \\ \langle US\rangle[2] : \texttt{ist}\,(politics, conservative(Powell)) \\ \langle US\rangle[1] : \neg\texttt{ist}(politics, \texttt{ist}(republican, \\ \qquad\qquad \neg leftist(Powell))) \end{array}$$

Here we assume that $\langle US\rangle[1]$ and $\langle US\rangle[2]$ are the only worlds generated at this stage by the tableau calculus.

To verify that this situation is indeed satisfiable, we can apply Neg-entering to the last contextual formula and thus introduce a new world $\langle US, politics\rangle[3]$ and the contextualized formula

$$\langle US, politics\rangle[3] : \neg\texttt{ist}\,(republican, \neg leftist(Powell))$$

Thus, in $\Sigma(\langle US, politics, republican\rangle)$ there should be a world where $\neg leftist(Powell)$ does not hold. We call this "new" world 6 and apply negative entering again to obtain:

$$\langle US, politics, republican\rangle[6] : \neg\neg leftist(Powell)$$

We continue with dneg to get:

$$\langle US, politics, republican\rangle[6] : leftist(Powell)$$

We may also go back to $\langle US\rangle[1]$ and check whether conservative and leftist are compatible by applying at first Pos-entering and then β to the implication. In one branch we get

$$\langle US, politics\rangle[3] : \neg conservative(Powell)$$

which is clearly contradictory with the knowledge gathered so far and makes it possible to discard the branch. However the other branch leads to

$$\langle US, politics\rangle[3] : \neg leftist(Powell)$$

which does not lead to a contradiction: the general context of US politics is a different from the more specific context of republican politics in US. We use a notion of contradiction which is sensible to the context:

Definition 4 *A branch is* closed *if two contextualized literals $\langle \vec{c}_n[m] : p\rangle$ and $\langle \vec{c}_n[m] : \neg p\rangle$ are both present in the branch for the same contextual prefix $\vec{c}_n[m]$. It is* completed *if all rules have been applied. It is* open *if it is completed and not closed.*

- `initialize` $\mathcal{B} := \{\vec{c}[0] : \Phi\}$
- `repeat`
  - `if` $\mathcal{B}$ contains $\vec{c}_n[m] : p$ and $\vec{c}_n[m] : \neg p$ for some prefix and some propositional letter `then stop` with failure;
  - `else if` $\mathcal{B}$ is completed `then stop` with SAT;
  - `else select` a formula $\vec{c}_n[m] : \psi$ not yet processed;
    - `mark` the formula as processed unless it is a $\vec{c}_n[m]$: `ist` (c_A, φ) which has not been applied to all $\vec{c}_n \circ c_A[k]$ present in $\mathcal{B}$;
    - `if` the (β) rule branch the tableau `then choose` nondeterministically a branch;
    - `add` the consequent formulae to $\mathcal{B}$;
    - `if` a new prefix $\vec{c}_n \circ c_A[k]$ has been introduced `then mark` all prefixed formulae of the form $\vec{c}_n[m]$: `ist` (c_A, φ) as not yet processed.

Figure 3: Algorithm CXT-SAT

Definition 5 *A tableau is closed if all branch are closed. It is open if at least one branch is open.*

Definition 6 *A* validity proof *for the formula Φ for the initial context $\vec{c}_i$ is the closed tableau starting with $\langle \vec{c}_i[0] : \neg\Phi \rangle$. A* satisfiability proof *for Φ is an open branch of the tableau starting with $\langle \vec{c}_i[0] : \Phi \rangle$.*

Intuitively, to prove that Φ is valid we try to construct a counter model for it and if we fail (the tableau is closed) we conclude that it is valid. Thus, we have an effective proof theory and a nondeterministic algorithm for the satisfiability of Φ can be easily derived (Alg. CXT-SAT). For a deterministic one, simply add backtracking at choice points as in (Fitting 1990).

Completeness and Complexity

Soundness and completeness of the calculus can be proved with an almost standard procedure (Fitting 1983; Massacci 1994).

Theorem 1 *A formula Φ is satisfiable for the context $\vec{c}_i$ iff the tableau starting with $\{\vec{c}_i[0] : \Phi\}$ is open.*

Proof. We sketch the completeness part. Start with $\{\vec{c}_i[0] : \Phi\}$ and apply a systematic procedure to the tableau. If it does not close, one can choose an open branch and construct a model.

The key idea is to use the labels $\vec{c}_n[m]$ occuring in the branch as possible worlds and to define the assignments of propositional variables so that $w(\vec{c}_n[m])(p) = true$ iff $\langle \vec{c}_n[m] : p \rangle$ is present in the branch. This can be safely done since for no $p \in \mathcal{P}$ both $\langle \vec{c}_n[m] : p \rangle$ and $\langle \vec{c}_n[m] : \neg p \rangle$ are present (otherwise the branch would be closed). Then, let Σ maps each sequence of contexts $\vec{c}_n$ into the set of valuations $w(\vec{c}_n[m])$ built so far.

Finally one exploits the completion of the branch to shown that if $\langle \vec{c}_n[m] : \phi \rangle$ is present in the branch then $\vec{c}_n, w(\vec{c}_n[m]) \models \phi$. For instance if $\langle \vec{c}_n[m] : \phi \wedge \psi \rangle$ is present in the branch both $\langle \vec{c}_n[m] : \phi \rangle$ and $\langle \vec{c}_n[m] : \psi \rangle$ must be present and the claim follow by a induction.

Hence $\vec{c}_i, w(\vec{c}_i[0]) \models \Phi$ and thus Φ is satisfiable. $\square$

Now we are left with the following...

Question 3 *Which is the complexity of CXT-SAT?*

The problem is clearly NP-hard since it subsumes propositional satisfiability (Cook 1971), and therefore we only need NP membership.

So let Φ be the formula to be checked and C, P and L be respectively the number of `ist` (), propositional connectives and different literals occuring in Φ.

Lemma 2 *Alg. CXT-SAT generate at most $O(C^3)$ different prefixes $\vec{c}_n[m]$ whichever branch is chosen.*

Proof. At first observe that any application of the entering rules decrease the number of `ist` () modalities present in the new formula. Hence the length of each sequence extending the initial $\vec{c}_i$ is bounded by C.

Next, for each sequence $\vec{c}_i \circ \langle c_{i+1}, \ldots, c_n \rangle$ present in the branch, there must be at least one sub-formula of the form `ist` $(c_{i+1}, \ldots,$ `ist` $(c_n, \ldots))$ occuring in Φ, where each `ist` () may be either prefixed by a negation or not. Since the number of `ist` () subformulae is bounded by C, so is the number of different sequences.

Third, given a sequence $\vec{c}_i \circ \langle c_{i+1}, \ldots, c_n \rangle$ we must prove that for each j between $i+1$ and n the number of different prefixes $\vec{c}_j[m]$ is bounded by C. The only rule which introduces new prefixes is negative entering (Fig. 2). Note that from the generating formula $\vec{c}_j[m]$: $\neg$`ist` (c_{j+1}, ϕ) to the new formula $\vec{c}_j \circ c_{j+1}[k] : \neg\phi$, the information about the generating world is discarded and only the information about the sequence $\vec{c}_j$ is kept. Thus, no matter how many worlds we generated at step j, at step $j+1$ we need a different $\neg$`ist` (c_{j+1}, ϕ) subformula of Φ to create a new world. These subformulae are also bounded by C and hence the claim. $\square$

Now, for every prefix $\vec{c}_n[k]$ we may have to reduce some propositional connectives i.e. $O(P)$ times $O(C^3)$. After this stage, when everything has been broken down into prefixed literals we must check for consistency each prefix $\vec{c}_n[m]$ i.e.check whether there are two contradictory literals and this takes $O(C^3 \times L)$. So a nondeterministic upper bound of $O(|\Phi|^4)$ follows by a simple multiplication:

Theorem 3 *Propositional logic of context is NP-complete and satisfiability can be checked in nondeterministic polynomial time* $O(|\Phi|^4)$.

One may wonder why this proof cannot be applied to modal logics. Indeed, the first two part of the proof of Lemma 2 hold for modal logics. The third step fails: the information about the generating world matters.

Comparison with Modal Logic

The logic of context, in its propositional form, can be seen as a normal multi modal logic: just replace $\texttt{ist}\,(c, \phi)$ with $[c]\phi$. Therefore one would have expected also the same complexity. On the contrary, modal logics for knowledge and belief are PSPACE-complete in the multi-modal case (Ladner 1977; Halpern & Moses 1992). It is therefore worth to analyze further the differences between these logics.

In the sequel we assume some knowledge of modal logic (Fagin *et al.* 1995; Halpern & Moses 1992). As mentioned, the language is (practically) the same, whereas the semantics is based on *Kripke models*, that is pairs $\langle W, \mathcal{R}_\mathcal{C}\rangle$ where W is a set of propositional valuation (possible worlds) and $\mathcal{R}_\mathcal{C}$ a family of relations over W. One may obtain different logics by varying $\mathcal{R}_\mathcal{C}$. In the sequel we focus on the simple logic K_n (no restriction on $\mathcal{R}_\mathcal{C}$).

Definition 7 *Let* $\langle W, \mathcal{R}_\mathcal{C}\rangle$ *be a Kripke model,* w *a valuation in* W*, then the entailment relation* $w \models \phi$ *is s.t.*

- $w \models p$ *iff* $w(p) = true$ *where* $p \in \mathcal{P}$*;*
- $w \models \phi \wedge \psi$ *iff* $w \models \phi$ *and* $w \models \psi$*;*
- $w \models \neg\phi$ *iff* $w \not\models \phi$*;*
- $w \models [c]\phi$ *iff for every* w^* *such that* $w\mathcal{R}_c w^*$ *it is* $w^* \models \phi$ *where* $\mathcal{R}_c \in \mathcal{R}_\mathcal{C}$

A formula Φ is *satisfiable* iff there is a Kripke model $\langle W, \mathcal{R}_\mathcal{C}\rangle$ such that for some $w \in W$ it is $w \models \phi$.

In the sequel we use the term *objective knowledge* of a world (either in a layered or a Kripke model) to denote the truth values it assigns to propositional letters. For instance in Fig. 4 worlds $\langle 1\rangle$ and $\langle 2\rangle$ differ for the objective knowledge of p.

The key part of the PSPACE-hardness proof in (Halpern & Moses 1992) is to show that in multi modal logic one can represent a complete tree of depth m with a short formula, polynomial in m. In case of Fig. 4 we can reformulate this problem as follows:

Question 4 *Can a formula say that worlds* $\langle 1\rangle$ *and* $\langle 2\rangle$ *generate a pair of worlds (with different objective knowledge) without being repeated twice?*

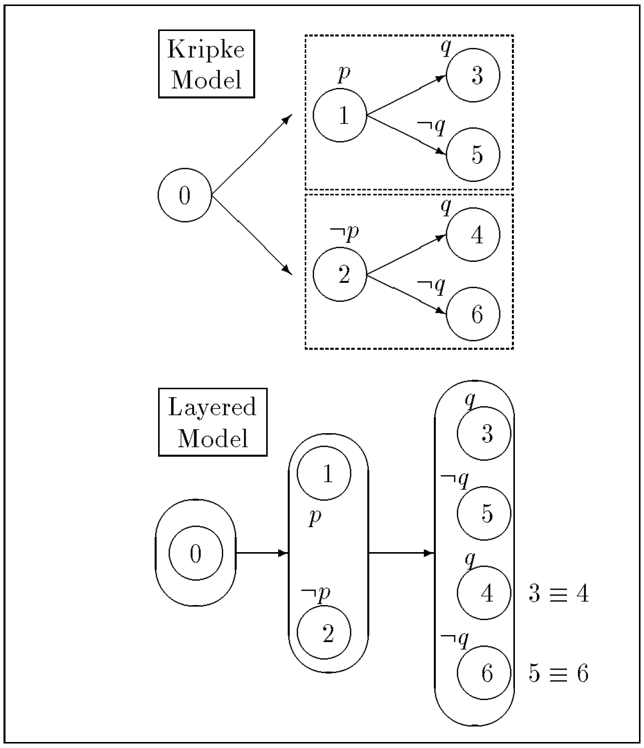

Figure 4: Kripke and Layered Models

Question 5 *Can a formula distinguish the subtree rooted in world* $\langle 1\rangle$ *from that rooted in world* $\langle 2\rangle$ *by looking locally at those worlds?*

If both answers are yes, we can represent a tree-like model with a formula of polynomial size.

Fact 1 *To represent branching either in the logic of context or in the modal logic* K *(replacing* $\texttt{ist}\,(c, \phi)$ *with* $[c]\phi$*) we can put in the world* $\langle 0\rangle$ *the formula*

$$\Phi_{\text{branch}} \quad = \quad \texttt{ist}\,(c, \neg\texttt{ist}\,(c, q) \wedge \neg\texttt{ist}\,(c, \neg q))$$

One can use the tableau to verify it: start with $\vec{c}_i[0] : \Phi_{\text{branch}}$ and assume that the context prefixes $\vec{c}_i \circ c[1]$ and $\vec{c}_i \circ c[2]$ were already present.

So question 4 does not discriminate enough.

Fact 2 *Modal logics can distinguish the subtrees with*

$$\Phi_{\text{local-inherit}} \quad = \quad [c]\,((p \rightarrow [c]p) \wedge (\neg p \rightarrow [c]\neg p))$$

but logic of context cannot.

The key point is that we can use $p \rightarrow [c]p$ to express a *local* constraints: once we arrive at $\langle 1\rangle$, then all worlds accessible from it will inherit p, without touching the worlds reachable through $\langle 2\rangle$. In the logic of context we have *global* inheritance: all worlds reachable in two

c-steps will be affected, no matter if reached through $\langle 1 \rangle$ or $\langle 2 \rangle$. Hence we can distinguish logics with:

$$\Phi_{\text{DIFF}} \doteq \neg[c]p \wedge \neg[c]\neg p \wedge \Phi_{\text{branch}} \wedge \Phi_{\text{local-inherit}}$$

Proposition 4 *The formula Φ_{DIFF} is satisfiable in modal logic K_n but not in logic of context.*

In the layered model shown in Fig. 4 this means that $\langle 3 \rangle$ and $\langle 4 \rangle$ are identical: both satisfy q and both can satisfy either p or $\neg p$ (but clearly cannot satisfy them together and thus Φ_{DIFF} is unsatisfiable).

At this stage one may wonder if the same trick used to prove PSPACE-completeness of $S5_n$ in (Halpern & Moses 1992) works here: replace $[c]$ with $[c_A][c_B]$ and check the satisfiability of Φ^{S5}_{DIFF} in the logic of context. One can use the tableau to check that

Proposition 5 *The formula Φ^{S5}_{DIFF} is satisfiable in modal logic $S5_2$ but is not in logic of context.*

Hence we may conclude with the following property:

Theorem 6 *Modal logic is exponentially more succinct than logic of context.*

Proof. In the modal logics for knowledge and belief $K_n \dots S4_n$ (and also $S5_n$ for $n \geq 2$) one can represent a Kripke model with *at least* 2^m worlds with a formula of size only $O(m^2)$, generalizing Φ_{DIFF} to tree of depth m as in (Ladner 1977; Halpern & Moses 1992). On the contrary, in the logic of context, each formula Φ can be satisfied in a layered model with *at most* $O(|\Phi|^3)$ worlds by Lemma 2. Thus, to represent structures with at least 2^m worlds one needs a formula almost of the same size i.e. $O(2^{m/3})$). $\square$

Still, the relationship between modal logics and logic of context is much subtler. Intuitively we may characterize a world by...

1. its objective knowledge,
2. the introspection steps necessary to reach it,
3. the particular path used to reach it.

Logic of context uses only the first two information to distinguish a pair of worlds whereas modal logic uses all three. Therefore logic of contexts can be seen as an *approximation* of modal logics K_n which takes into account only the steps of introspection. For instance in the mono-modal case (one context) we only look if we need one, two, three etc. steps of introspection.

Conclusions

We can summarize new and old results about the complexity and compactness of logic of context and modal logics (ML systems) as follows:

Tractability of Reasoning: logic of context is NP-complete whereas modal logics and ML systems are PSPACE-complete. Logic of context can be seen as an approximation of modal logic where only the steps of introspection does matter.

Compactness of Representation: the logic of context can store compactly non-theorems for a group of theories while modal logics (and ML systems) can represent in compact form such a tree *plus* local inheritance of theorems. Thus modal logic can be exponentially more succinct than logic of context.

At this stage we may look back to intelligent systems and ask whether these technical results can provide an indication for the choice of a formalism in practice.

Of course, there are many different viewpoints that must be taken into account: the expressiveness of the language, the possibility to cope with limited omniscience etc. In this respect we refer to Guha (1991a), McCarthy (1993) and Buvač et al. (1995) for the logic of context, to Giunchiglia et al. (1994) for Multi Language Systems and to (Fagin *et al.* 1995; Halpern & Moses 1992) for modal logics.

Still, in the quest for intelligent *and* effective systems some indications can be drawn:

- if positive knowledge is the main (and overwhelming) component then logic of context fits better;
- whenever positive and negative knowledge are mixed we may choose the logic of context if global inheritance of positive knowledge between theories is the main requirement;
- complex structures with negative knowledge and local inheritance of positive knowledge suit better modal logics since storing same information with logic of context may lead to an exponential blow up.

Finally there is also the possibility to use the tableau for the logic of context as a sound approximation for the satisfiability of modal logics.

As a conclusion, we may rephrase McCarthy's statement from a computational perspective:
"Modality yes, logic of context... sometimes".

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